Homework 6

Due 2/18/2010

1. [10 points + 5 extra credit points] Consider a metallic substance. We keep the spirit of the free electron approximation, but employ a slightly more general approach, by assuming a density of states function $D(\epsilon)$ valid for $\epsilon \approx \epsilon_F$ without specifying what the function really is except that $D(\epsilon_F)$ is a finite number. Our only other assumption is that $T \ll T_F$.
   a. The conservation of the electron number N means

   $$\int_0^{\epsilon_F} D(\epsilon)d\epsilon = \int_0^{\infty} D(\epsilon)f(\epsilon,\mu,T)d\epsilon = N$$

   Note that in this problem the Fermi Dirac function $f(\epsilon,T)$ is treated as a function of three variables $f(\epsilon,\mu,T)$, purely for a mathematical reason. Physically, $\mu = \mu(T,N)$.

   Show that this equation can be re-written as

   $$\int_{\epsilon_F}^{\mu} D(\epsilon)d\epsilon = \int_0^{\infty} D(\epsilon)\big(f(\epsilon,\mu,T=0) - f(\epsilon,\mu,T)\big)d\epsilon$$

   Note that $f(\epsilon,\mu,T=0)$ is a purely mathematical construct, which is a "step-down" function at $\epsilon = \mu(T)$, *different* from the Fermi Dirac function at $T = 0$, which is a step-down function at $\epsilon = \mu(T=0) = \epsilon_F$.

   b. Show that the function $f(\epsilon,\mu,T=0) - f(\epsilon,\mu,T)$ is an odd function of $\epsilon - \mu$. Also, show that it is exponentially small when $|\epsilon - \mu| \gg k_B T$, meaning that it is appreciably different from 0 only when $|\epsilon - \mu| < \sim k_B T$.
   c. Now, assume that $\mu \approx \epsilon_F$ at all temperatures of interest ($\ll T_F$). Using the results of a and b, and the Taylor expansion $D(\epsilon) \approx D(\mu) + D'(\mu)(\epsilon - \mu)$, show that, to leading order,

   $$\mu \approx \epsilon_F - \frac{\pi^2}{6}\frac{D'(\epsilon_F)}{D(\epsilon_F)}\,(k_B T)^2$$

   Use the iterative method. You can use $\int_0^{\infty} dx\,\frac{x}{e^x+1} = \frac{\pi^2}{12}$ without proof.

   d. Consider the case when $D(\epsilon) \propto \epsilon^{\alpha}$. Show that

   $$\mu \approx \epsilon_F\left(1 - \alpha\frac{\pi^2}{6}\left(\frac{T}{T_F}\right)^2\right)$$

e. What is the value of α for the free electron dispersion in 1 dimension, 2 dimensions, and 3 dimensions? Calculate the finite temperature correction to μ, to the $\left(\frac{T}{T_F}\right)^2$ order in each case.

f. [Extra credit: 5 points] Employing the same techniques (i.e. using $f(\epsilon,\mu,T) = \{f(\epsilon,\mu,T) - f(\epsilon,\mu,T=0)\} + f(\epsilon,\mu,T=0)$, $D(\epsilon) \approx D(\mu) + D'(\mu)(\epsilon-\mu)$, and the method of iteration), and using the above result $\mu \approx \epsilon_F - \frac{\pi^2}{6}\frac{D'(\epsilon_F)}{D(\epsilon_F)}(k_B T)^2$, show that the total energy is given as the following:

$$E = \int_0^\infty d\epsilon \overset{\epsilon}{} D(\epsilon) f(\epsilon,\mu,T) \approx E(T=0) + \frac{\pi^2}{6}(k_B T)^2 D(\epsilon_F)$$

Now, consider the case of free electrons in 3 dimensions. Using the fact that $E(T=0) = \frac{3}{5}N\epsilon_F$ (last homework) and $D(\epsilon_F) = \frac{3N}{2\epsilon_F}$ (lecture), show that

$$E \approx \frac{3}{5}N\epsilon_F + \frac{\pi^2}{4}N\epsilon_F\left(\frac{T}{T_F}\right)^2$$

and that

$$C_V \approx \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F}\right)$$

2. [5 points] Kittel 6.4

3. [5 points] Kittel 6.5

4. [5 points] Kittel 6.6

5. [5 points] Kittel 6.9